The complexity of shortest path and dilation bounded interval routing ☆

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Abstract

Interval routing is a popular compact routing method for point-to-point networks which found industrial applications in novel transputer routing technology (May and Thompson, Transputers and Routers: Components for Concurrent Machines, Inmos, 1991).

Recently much effort is devoted to relate the efficiency (measured by the dilation or the stretch factor) to space requirements (measured by the compactness or the total number of memory bits) in a variety of compact routing methods (Eilam, Moran and Zaks, 10th International Workshop on Distributed Algorithms (WDAG), Lecture Notes in Computer Science, vol. 1151, Springer, Berlin, 1996, pp. 191–205; Fraigniaud and Gavoille, 8th Annual ACM Symp. on Parallel Algorithms and Architectures (SPAA), ACM Press, New York, 1996; Gavoille and Pérennes, Proc. SIROCCO'96, Carleton Press, 1996, pp. 88–103; Kranakis and Krizanc, 13th Annual Symp. on Theoretical Aspects of Computer Science (STACS), Lecture Notes in Computer Science, vol. 1046, Springer, Berlin, 1996, pp. 529–540; Meyer auf der Heide and Scheideler, Proc. 37th Symp. on Foundations of Computer Science (FOCS), November 1996; Peleg and Upfal, J. ACM 36 (1989) 510–530; Tse and Lau, Proc. SIROCCO'95, Carleton Press, 1995, pp. 123–134). We add new results in this direction for interval routing.

For the shortest path interval routing we apply a technique from Flammini, van Leeuwen and Marchetti-Spaccamela (MFCS'95, Lecture Notes in Computer Science, vol. 969, Springer, Berlin, 1995, pp. 37–49) to some interconnection networks (shuffle exchange (SE), cube connected cycles (CCC), butterfly (BF) and star (S)) and get improved lower bounds on compactness in the form $\Omega(n^{1/2-\varepsilon})$, any $\varepsilon > 0$, for SE, $\Omega(\sqrt{n/\log n})$ for CCC and BF, and $\Omega(n(\log\log n/\log n)^5)$ for S, where n is the number of nodes in the corresponding network. Previous lower bounds for these networks were only constant (Fraigniaud and Gavoille, CONPAR'94, Lecture Notes in Computer Science, vol. 854, Springer, Berlin, 1994, pp. 785–796).

For the dilation bounded interval routing we give a routing algorithm with the dilation $\lceil 1.5D \rceil$ and the compactness $O(\sqrt{n \log n})$ on n-node networks with the diameter D. It is the first non-trivial upper bound on the dilation bounded interval routing on general networks. Moreover, we construct a network on which each interval routing with the dilation $1.5D - 3$ needs the compactness at least $\Omega(\sqrt{n})$. It is an asymptotical improvement over the previous lower bounds in Tse and Lau (Proc. SIROCCO'95, Carleton Press, 1995, pp. 123–134) and it is also better than independently obtained lower bounds in Tse and Lau (Proc. Computing: The Australasian Theory Symp. (CATS'97), Sydney, Australia, February 1997).

Keywords: Compact routing; Dilation bounded interval routing; Complexity

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1. Introduction

Interval routing is an attractive space-efficient routing method for point-to-point communication networks. Interval routing was introduced in [17] and generalized in [21]. It has found industrial applications in the C104 Router Chip used in the INMOS T9000 Transputer design [13] and has been exploited in fault-tolerant parallel networks [23].

Interval routing is based on compact routing tables, where the set of nodes reachable via outgoing links is represented by interval labels. By the compactness we measure the maximum number of interval labels per link. By the dilation we measure the length of the longest routing path in the network.

Most of the previous work was oriented towards optimal (shortest path) interval routing. Several classes of networks have optimal 1-IRS (i.e., routing schemes using up to 1 interval per link) [22]. But there are also networks without optimal 1-IRS [7, 6, 18]. To overcome this inefficiency, multi-label routing schemes were introduced. General n-node networks can be optimally routed with $\lfloor \frac{n}{2} \rfloor$-IRS. However, when no specific assumption about the network topology is made, optimal interval routing does not significantly reduce the bound $\lfloor \frac{n}{2} \rfloor$. In [4], a technique for proving lower bounds on the compactness was developed and it has been used in [7] to construct n-node networks for which each optimal k-IRS requires $k = \Theta(n)$. A similar result for random networks was obtained in [4].

For certain regular and symmetric networks (such as hypercubes or tori), optimal k-IRS exist for small k [2, 6]. A natural question arises whether there are also optimal k-IRS for small k for other well-known interconnection networks, such as shuffle exchange (SE), cube connected cycles (CCC), butterfly (BF) and star networks (S). In [7], it was proved that these networks have no optimal 1-IRS. We use a technique (derived from one in [4]) for obtaining lower bounds on compactness for the optimal IRS on arbitrary networks. We apply this technique to some interconnection networks, obtaining lower bounds on the compactness in the form $\Omega(n^{\frac{1}{2}-\varepsilon})$, any $\varepsilon > 0$, for SE, $\Omega(\sqrt{n/\log n})$ for CCC and BF, and $\Omega(n(\log\log n/\log n)^5)$ for S, where n is the number of nodes in the corresponding network.

Recently, much effort has been devoted to relate the efficiency (measured by the dilation) to space requirements (measured by the compactness). Each network has a 1-IRS with the dilation bounded by $2D$, where D is the diameter of the network [17]. However, there are also networks having long dilation for each 1-IRS. For n-node networks the lower bound for k-IRS with the dilation $1.75D-1$ was $k \geqslant 2$ [18], with the dilation $1.25D-1$ it was $k \geqslant 3$ [19] and with the dilation $[(2k+1)/2k]D-1$ and $[(6k+1)/6k]D-1$ it was $k=\Omega(\sqrt[3]{n})$ and $k=\Omega(\sqrt{n})$, respectively [19]. The basic question is whether there are interval routing schemes for arbitrary networks attaining short dilation with a reasonable small compactness. We answer this question negatively [1] by constructing an n-node network with the diameter D for which each routing scheme with the dilation $1.5D-3$ needs the compactness $\Omega(\sqrt{n})$. Moreover, we give a routing algorithm with the dilation $\lceil 1.5D \rceil$ and the compactness $O(\sqrt{n \log n})$. It is the first nontrivial upper bound for the dilation bounded interval routing on general networks.

The paper is organized as follows. Section 1 presents the model together with the description of the routing problem under discussion. Section 2 is devoted to the shortest path routing in interconnection networks. In Section 2.1 we introduce a technique for obtaining lower bounds on the compactness for the optimal IRS on arbitrary networks. In Sections 2.2–2.5 we apply this technique to SE, CCC, BF and S networks. In Section 3 we investigate interval routing with bounded dilation. In Sections 3.1 and 3.2 we give a new lower bound technique together with lower and upper bounds for dilation bounded routing on general networks. In Sections 3.3 and 3.4 we relate the compactness to the dilation for multiglobe networks.

1.1. Definitions

We assume a point-to-point asynchronous communication network. The network topology is modeled by a simple connected graph $G=(V,E)$, where V is a set of vertices (or processors) and E is a set of edges (or bidirectional links) in G. Assume $|V|=n$. The diameter of G is denoted as $D(G)$. Given a vertex $v \in V$, by $I(v)$ we denote the set of arcs outgoing from v.

Interval routing is based on a suitable labeling scheme for the vertices and edges in a graph. A vertex label is an element of the set $\{1,\dots,n\}$ and an arc label is a cyclic interval $[a,b]$ with $a,b \in \{1,\dots,n\}$. An *interval labeling scheme* (shortly ILS) of G is a scheme, where

- a vertex labeling is an assignment of unique labels to vertices of V;
- for each vertex $v \in V$, an edge labeling is an assignment of disjoint intervals to arcs $e \in I(v)$.

Given an ILS on G, a routing performs in the following way. Let a message destinated for a vertex w currently reach some vertex u, $u \neq w$. Determine an arc $e=(u,v) \in I(u)$ such that a label of w belongs to an interval assigned at u to e. Transmit a message from u to v.

[1] The same conclusion was independently obtained by Tse and Lau [20]. However, they proved weaker results of compactness $\Omega(\log n)$ for dilation $1.5D-O(1)$ and of compactness $\Omega(\sqrt{n})$ for dilation $1.25D-O(1)$.

If the edge labeling assigns to each arc at most k intervals, the scheme is called k interval labeling scheme (shortly k-ILS). If the routing strategy guarantees that the messages always arrive at their destination, then k-ILS is termed as k-IRS (k *interval routing scheme*).

Formally, $\rho = (\lambda, R)$ is a k-ILS, if
1. a mapping $\lambda : V \mapsto \{1, \ldots, n\}$ is a bijection,
2. a mapping $R : V \times E \mapsto 2^V$ satisfies
   - (completeness) for $v \in V : V - \{v\} \subseteq \bigcup_{e \in I(v)} R(v, e)$;
   - (determinism) for $v \in V,\ e_1, e_2 \in I(v),\ e_1 \neq e_2 : R(v, e_1) \cap R(v, e_2) = \emptyset$;
   - (compactness) for $v \in V,\ e \in I(v) : \{\lambda(u) | u \in R(v, e)\}$ forms up to k compact cyclic intervals.

     Note that $R(v, e)$ must not be specified for $v \in V,\ e \notin I(v)$. In the specification of a k-IRS, the completeness condition is replaced by
   - (validity) for $v_1, v_l \in V$ there are $v_2, \ldots, v_{l-1} \in V$ s.t. $v_l \in R(v_i, (v_i, v_{i+1}))$ for $1 \leqslant i \leqslant l - 1$;

Given a graph G and a k-IRS ρ on G, a *routing path system* (for ρ on G) is the set of routing paths for all pairs of vertices in V. The *dilation*, denoted as $dil(G, \rho)$, is the length of the longest path in the routing path system for ρ on G. A k-IRS is called *optimal*, if all paths in the routing path system are the shortest ones. A k-IRS is called α-*bounded* (shortly (k, α)-IRS) if the dilation $dil(G, \rho)$ is limited by α. For optimal routing the *compactness* of G is the minimum k such that there is a k-IRS on G. For α-bounded routing the compactness of G denotes the minimum k such that there is a (k, α)-IRS on G.

Notations: we use the following symbols: $\#_c S$ – for the number of occurences of a symbol c in a sequence S; $\oplus$ and $\ominus$ – for the addition and subtraction modulo some number n.

2. Shortest path interval routing

This section is devoted to the shortest path interval routing for some interconnection networks. We present a technique for obtaining a lower bound on the compactness for the shortest path interval routing on arbitrary graphs. It is derived from a technique in [4], which is also used in [7, 9]. Then, we apply this technique to certain interconnection networks (shuffle exchange, cube connected cycles, butterfly and star) to obtain asymptotical improvements over the previous constant lower bounds [7].

2.1. A lower bound technique for optimal routing

During this subsection, let $G = (V, E)$ be a simple connected graph with maximum degree Δ and $\rho = (\lambda, R)$ be an arbitrary optimal k-IRS of the graph G.

For a vertex $v \in V$ and an arc $e \in I(v)$, denote $S(v, e)$ the subset of vertices $w \in V$ which can be reached optimally from v over its outgoing arc e and $Z(v, e)$ the subset of vertices $w \in V$ such that every optimal path from v to w follows the outgoing arc e.

In the following theorem we present a lower bound on the number of intervals for an optimal interval routing scheme in G. The idea of the proof technique is based on the so called "wq–property": Given the graph G, we choose two disjoint sets of vertices W and Q such that for any distinct vertices $w_i, w_j \in W$ there is a vertex $v \in Q$ such that in any optimal routing scheme the messages sent by v to w_i and w_j are routed along different outgoing arcs.

Theorem 1. *Let G be a graph with maximum degree Δ and $\rho = (\lambda, R)$ be an optimal k-IRS of G. Let $Q = \{q_1, \ldots, q_l\}$ and $W = \{w_1, \ldots, w_m\}$ be disjoint vertex subsets of G satisfying wq-property, that means for $w_i, w_j \in W$, $w_i \neq w_j$, there is $v \in Q$ such that for each $e \in I(v)$ it holds $w_i \notin S(v, e)$ or $w_j \notin S(v, e)$. Then it holds*

$$k \geqslant \frac{m}{l\Delta}. \tag{1}$$

Proof. Assume $\rho = (\lambda, R)$ is an arbitrary optimal k-IRS for G. W.l.o.g assume that $\lambda(w_1) < \lambda(w_2) < \ldots < \lambda(w_m)$. Consider an arbitrary sequence P of all different pairs (v, e) with $v \in Q$, $e \in I(v)$ and denote as p the length of this sequence. Clearly, $p \leqslant l\Delta$. A matrix representation of ρ w.r.t. P and W is an $p \times m$ Boolean matrix $M(P, W)$ with $M(P, W)[i, j] = 1$ if $w_j \in R(v, e)$ for the ith pair (v, e) in P and $M(P, W)[i, j] = 0$ otherwise, for every $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant m$. Considering the ith row of M for any $1 \leqslant i \leqslant p$, the blocks of consecutive 1's separated by 0's in the cyclic row correspond to the intervals assigned to the arc e at v by the optimal k-IRS ρ, where the ith pair in P is (v, e). So the number of intervals is equal to the number of occurences of the pattern 10 in the cyclic row.

In each row of M there are at most k positions at which the pattern 10 matches the row cyclically. Hence, for each row in M (i.e., each pair $v \in Q$, $e \in I(v)$) it holds

$$\sum_{w_j \in W} (w_j \in R(v, e) \wedge w_{j \oplus 1} \notin R(v, e)) \leqslant k$$

and consequently

$$\sum_{v \in Q} \sum_{e \in I(v)} \sum_{w_j \in W} (w_j \in R(v, e) \wedge w_{j \oplus 1} \notin R(v, e)) \leqslant kp \leqslant kl\Delta. \tag{2}$$

On the other hand, provided that

- for each j, $1 \leqslant j \leqslant m$, there is $v \in Q$ such that for each $e \in I(v)$ it holds $w_j \notin S(v, e)$ or $w_{j \oplus 1} \notin S(v, e)$, and
- for each $w_j \in W$, $v \in Q$ there is unique $e \in I(v)$ satisfying $w_j \in R(v, e)$,

we conclude that for each column j in M there is at least one row such that the pattern 10 matches this row at position j, cyclically. Thus, for arbitrary column in M (i.e., for each $w_j \in W$) it holds

$$\sum_{v \in Q} \sum_{e \in I(v)} (w_j \in R(v, e) \wedge w_{j \oplus 1} \notin R(v, e)) \geqslant 1.$$

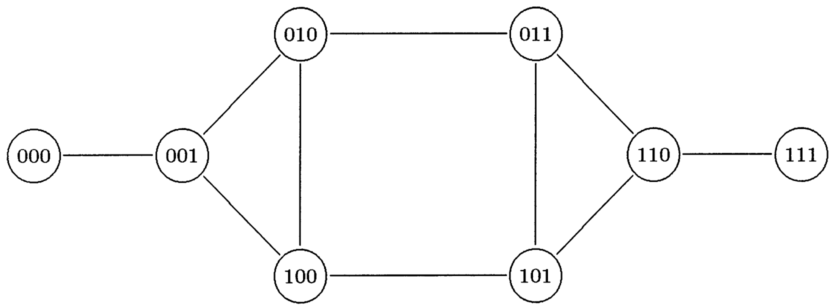

Fig. 1. Graph $SE(3)$.

Hence

$$\sum_{w_j \in W} \sum_{v \in Q} \sum_{e \in I(v)} (w_j \in R(v,e) \wedge w_{j \oplus 1} \notin R(v,e)) \geqslant m. \tag{3}$$

Combining inequalities (2) and (3) we get (1). $\square$

2.2. *A lower bound for shuffle exchange*

Let $\alpha = a_{d-1} \ldots a_1 a_0$ be a binary string. The left cyclic shift and the right cyclic shift operations on α are denoted as $L(\alpha)$ and $R(\alpha)$, respectively, and the shuffle operation $S(\alpha)$ is defined as $S(\alpha) = a_{d-1} \ldots a_1 \hat{a}_0$, where $\hat{a}_0 = 1 - a_0$.

The shuffle exchange graph of degree d (denoted as $SE(d)$) is a graph (V, E), where $V = \{u \mid u \in \{0,1\}^d\}$ and $E = \{(u,v) \mid R(u) = v$ or $L(u) = v$ or $S(u) = v\}$. An arc (u,v) is called R-arc, L-arc, S-arc if $R(u) = v$, $L(u) = v$, $S(u) = v$, respectively. For a path C in the $SE(d)$ let $\#_S C$ denote the number of S-arcs in this path (similarly we define $\#_L C$, $\#_R C$ and $\#_{L,R} C$).

The L and R operations do not change the number of ones in a binary string and S operation changes their amount by ± 1, therefore:

Proposition 1. *For any path C in the $SE(d)$ from u to v it holds $\#_S C \geqslant |\#_1 u - \#_1 v|$.*

It is convenient to represent vertices of the $SE(d)$ as binary strings with cursor denoting the least significant bit cyclically (see Fig. 1 for $SE(3)$). For example, $1111\underline{0}101 = 10111110$ (the operation of erasing the cursor is denoted by "$\mapsto$" i.e. $1111\underline{0}101 \mapsto 11110101$). Now instead of rotating the string cyclically to the left (to the right) it suffices to move the cursor to the right (to the left). Changing the bit pointed by the cursor corresponds to the shuffle operation.

Consider any path C in the $SE(d)$ from u to v and let $k = \#_R C \ominus \#_L C$. Manipulating the cursor we get from $u = u_{d-1} \ldots \underline{u_0} \mapsto u'$ to $v = v_{(d-1)\ominus k} \ldots \underline{v_0} v_{d-1} \ldots v_{0 \ominus k} \mapsto v'$. If $v'_i = v_{i \ominus k}$ differs from $u'_i = u_i$ then there must be a situation in which the cursor was pointing at the ith bit. We can formulate the following lemma:

Lemma 1. *Let C be a path in the $SE(d)$ from u to v and let $k=\#_R C \ominus \#_L C$. Let $x_0=0$ and $x_1 < \ldots < x_{t-1}$ be positions in which $u'=u$ and $v'=L^k(v)$ differ. The following inequality holds*:

$$\#_{L,R} C \geqslant d - \max_{i\in\{0,\ldots,t-1\}} (x_{i\oplus 1} \ominus x_i).$$

Moreover, if the equality holds, then there are no L-arcs or no R-arcs in the path C.

Proof. For each position x_i there must be a situation such that the cursor was pointing at the x_i-th bit. The set of all positions of the cursor is a cyclic interval, therefore at most $\max_{i\in\{0,\ldots,t-1\}}(x_{i\oplus 1}\ominus x_i)-1$ bits have not been pointed by the cursor. To change the position of the cursor one L or R operation is required and therefore

$$\#_{L,R} C \geqslant d - \max_{i\in\{0,\ldots,t-1\}} (x_{i\oplus 1} \ominus x_i).$$

If there are both L-arcs and R-arcs in C then at least one position becomes pointed twice and therefore the inequality is sharp. $\square$

Theorem 2. *For any $\varepsilon > 0$ there exists d_0 such that each optimal k-IRS for any $SE(d)$ with $d \geqslant d_0$ needs $k=\Omega(n^{\frac{1}{2}-\varepsilon})$ intervals.*

Proof. Let $d=2(m+1)^2+p-1$, where p is the smallest possible. Let A be the set of all suffixes a of strings of the form $(\{0,1\}^m1)^m$ such that the length of any a is different from $(m+1)i+1$ for any i. Let B be the set of all prefixes b of strings of the form $(1\{0,1\}^m)^m$ such that the length of any b is different from $(m+1)i+1$ for any i. Consider the following sets W and Q:

$$W=1^p(\{0,1\}^m1)^m0^m\underline{1}0^m(1\{0,1\}^m)^m,$$

$$Q=\bigcup_{a\in A}\{0^p0^{m(m+1)-|a|-1}\underline{0}a0^m00^m0^{m(m+1)}\}\cup\bigcup_{b\in B}\{0^p0^{m(m+1)}0^m00^mb\underline{0}0^{m(m+1)-|b|-1}\}.$$

Clearly, $|W|=2^{2m^2}$ and $|Q|=2\cdot(2^{m^2+1}-1)$. We need to show that W,Q satisfy the “wq-property” of Theorem 1. Consider w_1,w_2 from W, $w_1\neq w_2$. W.l.o.g. suppose that w_1 and w_2 differ somewhere to the left of the cursor. Then

$$w_1=1^pr_10q0^m\underline{1}0^ms_1\mapsto w_1',$$

$$w_2=1^pr_21q0^m\underline{1}0^ms_2\mapsto w_2'.$$

Choose the following v from Q,

$$v=0^p0^{|r_1|}\underline{0}q0^m00^m0^{|s_1|}\mapsto v'.$$

Take the following path from v to w_i. Move the cursor to the left until it reaches the same position as is the cursor position in w_i, and by the way change all bits in which w_i' and v' differ. We obtain a path of the length

$$\#_1 w_i - \#_1 v + d - |q| - m - 1.$$

Due to the Proposition 1, for any path C from v to w_i

$$\#_S C \geqslant \#_1 w_i - \#_1 v.$$

Therefore for any shortest path C from v to w_i it must hold

$$\#_{L,R} C \leqslant d - |q| - m - 1.$$

For arbitrary k the $L^k(w_i)$ does not contain $m+1$ consecutive zeros. If $x_0 = 0$ and $x_1 < \ldots < x_{t-1}$ are the positions in which $v = q0^{d-|q|}$ and $L^k(w_i)$ differ then

- If $x_i < x_{i\oplus 1} \leqslant d - 1 - |q|$ then $x_{i\oplus 1} \ominus x_i \leqslant m + 1$.
- If $x_i \leqslant d-1-|q|$ and either $x_{i\oplus 1} = 0$ or $x_{i\oplus 1} > d-1-|q$, then $x_{i\oplus 1} \ominus x_i \leqslant m+1+|q|$.
- If $x_i > d - 1 - |q|$, then $x_{i\oplus 1} \ominus x_i \leqslant |q| - 1$.

Therefore

$$\max_{i\in\{0,\ldots,t-1\}} (x_{i\oplus 1} \ominus x_i) \leqslant m + 1 + |q|$$

and using Lemma 1 we get

$$\#_{L,R} C \geqslant d - |q| - m - 1.$$

Therefore, for the shortest path it holds

$$\#_{L,R} C = d - |q| - m - 1$$

and from the second part of Lemma 1 it follows that there are no R-arcs or no L-arcs in C. In the case with no R-arcs at least $d - m$ cursor moves to the right are required, because w_i contains no $m+1$ consecutive zeros and first $m+1$ bits of v are zeros. It follows, that there is exactly one shortest path from v to w_1, which starts with R-arc and there is exactly one shortest path from v to w_2, which starts with S-arc, therefore the “wq-property” of W, Q from Theorem 1 is satisfied and the following bound holds on the number of intervals k necessary for any optimal k-IRS of $SE(d)$:

$$k \geqslant \frac{|W|}{\Delta|Q|} = \frac{2^{2\cdot m^2}}{3 \cdot 2 \cdot (2^{m^2+1} - 1)} > 2^{m^2-4}.$$

As $d = 2(m+1)^2 - 1 + p$, take $m = \lfloor \sqrt{\frac{d+1}{2}} \rfloor - 1$. Then

$$2^{m^2-4} = 2^{d(\frac{1}{2} - \mathrm{O}(d^{-\frac{1}{2}}))}$$

and therefore for any positive constant ε it holds

$$k = \Omega(n^{\frac{1}{2}-\varepsilon}). \qquad \square$$

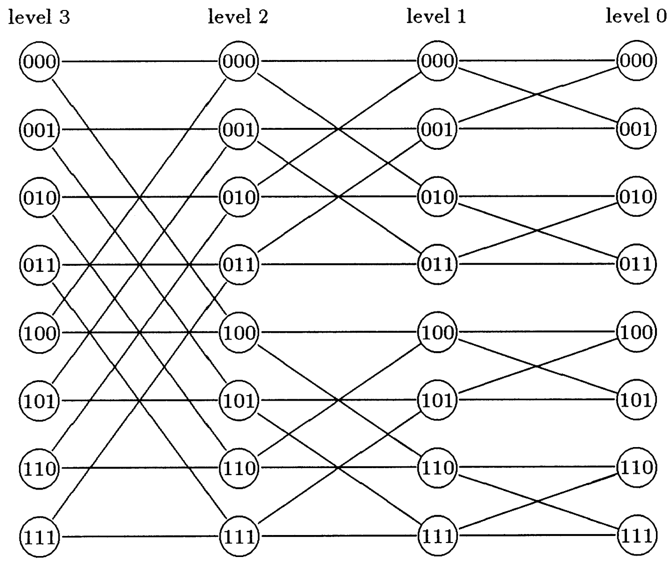

Fig. 2. Graph $BF(3)$.

2.3. A lower bound for butterfly

The butterfly graph of degree d (denoted as $BF(d)$) consists of $d+1$ levels, each level containing 2^d vertices, each of them labeled with unique binary string of the length d. An edge connects two vertices in $BF(d)$ if and only if they are in the consecutive ith and $(i+1)$st levels, respectively, and their labels are either equal or differ only in the ith bit (graph $BF(3)$ is shown in Fig. 2).

Let $\alpha = a_{d-1}\dots a_0$ be a binary string and x be a level $0 \leqslant x \leqslant d$. It is convenient to represent the vertex with given binary string α and the level x as binary string with cursor denoting the level: $a_{d-1}\dots a_x \triangledown a_{x-1}\dots a_0$. The arcs of $BF(d)$ graph corresponding to moving the cursor to the left, to the right are called L-arcs, R-arcs, respectively. The arcs corresponding to moving the cursor to the left, to the right and simultaneously changing the passed bit are called S_L-arcs, S_R-arcs, respectively.

Theorem 3. *Each optimal k-IRS for any $BF(d)$ needs $k = \Omega(\sqrt{n/\log n})$ intervals.*

Proof. Let $x = \lfloor \frac{d}{2} \rfloor$ and $y = \lceil \frac{d}{2} \rceil$. Consider disjoint sets W and Q:

$$W = \{0,1\}^x \triangledown \{0,1\}^y,$$
$$Q = \bigcup_{i=0}^{x-1} \{0,1\}^i \triangledown 0^{d-i} \ \cup \ \bigcup_{i=0}^{y-1} 0^{d-i} \triangledown \{0,1\}^i.$$

Clearly $|W| = 2^{x+y}$ and $|Q| = 2^x + 2^y - 2$. We need to show that W, Q satisfy the "wq-property" of the Theorem 1. Consider $w_1, w_2 \in W$, $w_1 \neq w_2$. W.l.o.g. suppose that

w_1 and w_2 differ somewhere to the left of the cursor. Then for some $|\alpha| \leqslant x-1$

$$w_1 = \alpha 0 s_1 \nabla r_1, \quad w_2 = \alpha 1 s_2 \nabla r_2.$$

Choose the following $v \in Q$

$$v = \alpha \nabla 0^{d-|\alpha|}.$$

Paths can be viewed as sequences of names of their arcs. If we take a path from v to w_i starting with L or S_L arc then removing arcs incident with vertices having the cursor more to the left than v we obtain a shorter path. If we take a path from v to w_1 starting with S_R-arc then we can make it shorter by replacing the first arc by R-arc and removing arcs incident with vertices having the cursor more to the left than v or at the same position as v. Similarly we can shorten a path from v to w_2 starting with R-arc. Therefore every shortest path from v to w_1 starts with R-arc and every shortest path from v to w_2 starts with S_R-arc and hence the "wq-property" is satisfied.

As a consequence of Theorem 1 it holds

$$k \geqslant \frac{|W|}{\Delta|Q|} = \frac{2^{x+y}}{4 \cdot (2^x + 2^y - 2)} \geqslant 2^{\lfloor \frac{d}{2} \rfloor - 3}$$

and thus

$$k = \Omega(\sqrt{n/\log n}). \qquad \square$$

2.4. A lower bound for cube connected cycles

Let $\alpha = a_{d-1} \ldots a_1 a_0$ be a binary string. Operations of shifting cursor cyclically to the left and to the right on α are denoted as $L(\alpha)$ and $R(\alpha)$, respectively, and the shuffle operation is defined as $S(\alpha) = a_{d-1} \ldots \hat{a}_x \ldots a_1 a_0$, where $\hat{a}_x = 1 - a_x$.

The cube connected cycles graph of degree d (denoted as $CCC(d)$, see Fig. 3 which shows graph $CCC(3)$) is obtained from a d-dimensional hypercube by replacing each vertex with a circle of length d. It consists of $d2^d$ vertices. Each vertex can be labeled with a binary string of length d and a cursor position in this string. An arc connects two vertices u, v if and only if v can be obtained from u by the means of cyclically shifting the cursor to the left or right (L- and R-arcs) or by changing the bit pointed by cursor (S-arcs).

The notation is as follows: having a binary string $\alpha = a_{d-1} \ldots a_1 a_0$ and a cursor position $0 \leqslant x < d$ the vertex is denoted as $a_{d-1} \ldots \underline{a_x} a_{x-1} \ldots a_1 a_0$.

Theorem 4. *Each optimal k-IRS for any $CCC(d)$ needs $k = \Omega(\sqrt{n/\log n})$ intervals.*

Proof. Let $d = 2x + 3 + a$, where $a \in \{2,3\}$. Consider following sets W and Q.

$$W = 0\{0,1\}^x 1^a \{0,1\}^x 0\underline{0}$$

$$Q = \bigcup_{i=0}^{x-1} 0\{0,1\}^i \underline{0} 0^{d-i-2} \ \cup \ \bigcup_{i=0}^{x-1} 0^{d-i-1} \underline{0} \{0,1\}^i 00.$$

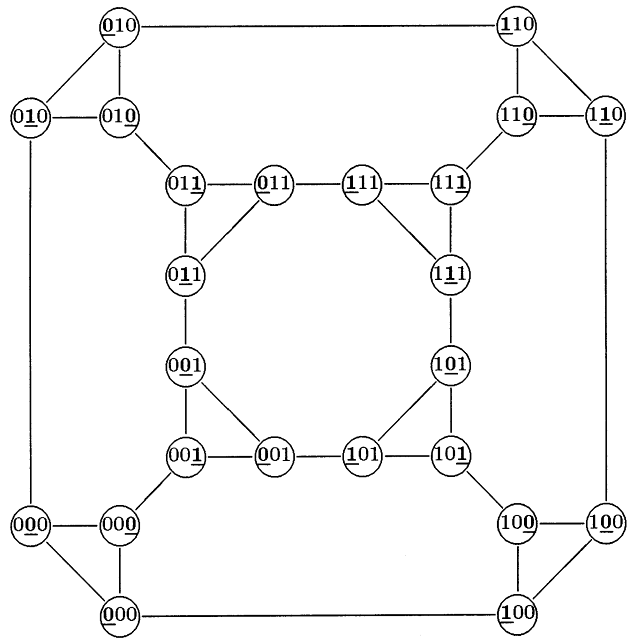

Fig. 3. Graph $CCC(3)$.

Clearly $|W| = 2^{2x}$ and $|Q| = 2(2^x - 1)$. We need to show, that W, Q satisfy the "wq-property" of Theorem 1. Consider w_1, w_2 from W, $w_1 \neq w_2$. W.l.o.g. suppose that w_1 and w_2 differ somewhere on the first $x+1$ positions from left. Then

$$w_1 = 0q0p_1 1^a r_1 0\underline{0}, \qquad w_2 = 0q1p_2 1^a r_2 0\underline{0}.$$

Choose the following v from Q:

$$v = 0q\underline{0}0^{d-|q|-2}.$$

Clearly no shortest path from v to w_1 starts with an S-arc. Thus it is sufficient to show that each shortest path from v to w_2 starts with an S-arc.

Take the following path from v to w_2. Shift the cursor to the right $d - |q| - 2$ times changing all bits in which v and w_2 differ. The length of this path is $d - |q| - 2 + H(v, w_2)$, where $H(u, v)$ denotes the Hamming distance between u and v, i.e. the number of bits in which they differ. Therefore for every optimal path C between v and w_2 it holds that $|C| \leqslant d - |q| - 2 + H(v, w_2)$. Clearly there must be at least $H(v, w_2)$ S-arcs in C and therefore there are at most $d - |q| - 2$ non-S arcs.

Because v and w_2 differ in $2 + x$-th bit then there must be a vertex u on the path C such that the cursor is on the position $2 + x$. Clearly at least $d - |q| - x - 4$ non-S arcs are needed to move the cursor from v to u and at least $2 + x$ of them to move the cursor from u to w_2. Thus there are exactly $d - |q| - 2$ non-S arcs in C and the only way to achieve this is when C consists exclusively from R- and S-arcs. Hence C must start with an S-arc.

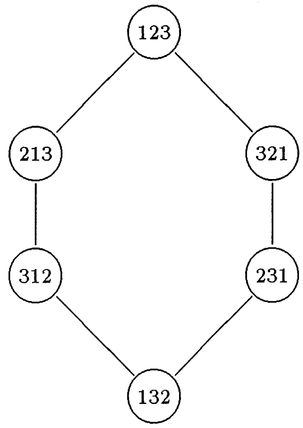

Fig. 4. Graph $S(3)$.

Now following Theorem 1 we get for the number of intervals

$$k \geqslant \frac{|W|}{\Delta|Q|} = \frac{2^{2x}}{3 \cdot 2 \cdot (2^x - 1)} > \frac{2^{x-1}}{3}.$$

As $x = \lfloor \frac{d-5}{2} \rfloor$ and $n = d \cdot 2^d$ it holds that

$$k = \Omega(\sqrt{n/\log\, n}). \qquad \square$$

2.5. A lower bound for star

The star graph of degree d (denoted as $S(d)$) has $d!$ vertices, each vertex being labeled by a unique permutation of the set $\{1,\dots,d\}$. An edge connects two vertices in $S(d)$ if and only if their labels can be mutually obtained by interchanging the first element with another one. Graph $S(3)$ is shown in Fig. 4.

Let $\mathscr{P}_d$ be the set of all permutations of $\{1,\dots,d\}$. By $\circ$ we denote the standard composition operation, where $(\pi \circ \xi)(x) = \pi(\xi(x))$ for some element x. By $(a_0,\dots,a_{k-1})$ we denote a *cycle*, i.e. a permutation π such that $\pi(a_i) = a_{i \oplus 1}$ and $\pi(x) = x$ for all other elements.

Formally, $S(d)$ is a graph $(\mathscr{P}_d, E)$, where $E = \{(u,v) \mid v = u \circ (1,j) \textit{ for } j \in \{2,\dots,d\}\}$.

Proposition 2. *Let $S(d) = (V,E)$. For each vertex $v \in V$ the projection $\mathscr{S}_v$ of the form $\mathscr{S}_v(w) = v^{-1} \circ w$ for $w \in V$ is an automorphism of $S(d)$ which maps v onto the identity permutation π_{id}.*

Proposition 3 (Akers, Harel and Krishnamurthy [1]). *The distance from a vertex u to the identity permutation π_{id} is*

$$d(u, \pi_{id}) = c + m - \begin{cases} 0 & \text{for } u(1) = 1, \\ 2 & \text{for } u(1) \neq 1, \end{cases}$$

where c is the number of cycles with the length at least two of u and m is the number of elements in these cycles (i.e., $m = |\{i \in \{1, \ldots, d\} \mid u(i) \neq i\}|$).

Theorem 5. *Each optimal k-IRS for any $S(d)$ needs $k = \Omega(n(\log\log n/\log n)^5)$ intervals.*

Proof. Consider disjoint sets of vertices W and Q

$$\begin{aligned} W &= \{(1,d) \circ (x_1, \ldots, x_{d-2}) \mid \{x_1, \ldots, x_{d-2}\} = \{2, \ldots, d-1\}\}, \\ Q &= \{(1,a) \mid a \in \{2, \ldots, d-1\}\}. \end{aligned}$$

Clearly, $|W| = (d-3)!$ and $|Q| = d-2$. We need to show that W, Q satisfy the "wq-property" of Theorem 1. Consider $w_1, w_2 \in W$, $w_1 \neq w_2$. Then there exist three different elements $a, b_1, b_2 \in \{2, \ldots, d-1\}$ such that $w_1(b_1) = w_2(b_2) = a$. Choose $v = (1,a) \in Q$. In order to show the "wq-property" we prove that for each arc e at v it holds either $w_1 \notin S(v,e)$ or $w_2 \notin S(v,e)$.

Following Proposition 2 it follows that the distance $d(w_i, v) = d(v^{-1} \circ w_i, \pi_{id})$. For $i \in \{1,2\}$ it holds $v^{-1} \circ w_i = (1,a) \circ (1,d) \circ (b_i, a, x_1, \ldots, x_{d-4}) = (1, d, a, x_1, \ldots, x_{d-4}, b_i)$ for some $x_1, \ldots, x_{d-4}$. Moreover $w_i(1) = d \neq a = v(1)$ and therefore following Proposition 3 we obtain $d(w_i, v) = d-1$.

Each arc e outgoing from v is of the form $e = (v, v \circ (1,l))$, where $1 < l \leqslant d$. Consider the edge $e = (v, v \circ (1,l))$, where $1 < l \leqslant d$ and $l \neq b_i$. We show that $d(w_i, v \circ (1,l)) = d > d-1 = d(w_i, v)$ and hence $w_i \notin S(v,e)$. Let w_i be of the form $(1,d) \circ (b_i, a, x_1, \ldots, x_{d-4})$, then $(v \circ (1,l))^{-1} \circ w_1 = (1,l) \circ (1,a) \circ (1,d) \circ (b_i, a, x_1, \ldots, x_{d-4})$. Because of the definition of W and Q there are only three possibilities for l: $l = x_j$ for some j, $l = a$ or $l = d$. Applying Propositions 2 and 3 we get for each case $d(w_i, v \circ (1,l)) = d$.

Because $b_1 \neq b_2$, for each arc e outgoing from v it holds either $w_1 \notin S(v,e)$ or $w_2 \notin S(v,e)$. Thus we can apply Theorem 1 to obtain

$$k \geqslant \frac{|W|}{\Delta|Q|} = \frac{(d-3)!}{(d-1)(d-2)} \geqslant \frac{(d-5)!}{6}$$

which gives

$$k = \Omega\left(n\left(\frac{\log\log n}{\log n}\right)^5\right). \qquad \square$$

2.6. Summary

In Table 1 we summarize results from Section 2 concerning lower bounds on the number of intervals needed for the shortest path interval routing for certain hypercube-like interconnection networks.

Table 1

Interconnection network	Compactness
Shuffle-exchange	$\Omega\left(n^{1/2-\varepsilon}\right)$
Cube-connected-cycles	$\Omega\left(\sqrt{\frac{n}{\log n}}\right)$
Butterfly	$\Omega\left(\sqrt{\frac{n}{\log n}}\right)$
Star graph	$\Omega\left(n\left(\frac{\log\log n}{\log n}\right)^5\right)$

3. Interval routing with bounded dilation

Recall that a k-IRS ρ of a graph $G=(V,E)$ is *α-bounded* if for every pair of vertices u,v from V the length of the routing path from u to v is limited by α (i.e. $dil(G,\rho)\leqslant\alpha$). Denote α-bounded k-IRS as (k,α)-IRS.

Note that usually the number of intervals is expressed in terms of the size of the graph and the length of the routing path is expressed in terms of the diameter of the graph.

The notion of α-boundary was studied in [16–19]. Each graph has $(1,2D)$-IRS [17] and can be optimally routed with $\lfloor\frac{n}{2}\rfloor$ intervals. Moreover, there are graphs for which $(1.75D-1)$-bounded routing requires at least 2 intervals [18] and $(1.25D-1)$-bounded routing at least 3 intervals [19]. The basic question is whether one can hope to find an interval routing scheme for arbitrary graphs with a short dilation and simultaneously with a reasonable small number of intervals. The main result of this section is a negative answer to this question, stating that there are graphs for which a routing with the dilation bounded by $1.5D-3$ needs $\Omega(\sqrt{n})$ intervals. We also show that $O(\sqrt{n\log n})$ intervals are sufficient for routing in arbitrary graphs with the dilation $\lceil 1.5D\rceil$.

The class of multiglobe graphs has been studied in connection with the lower bound proofs for dilation bounded interval routing in [18]. Exploiting techniques from subsection 3.1 introduced for general graphs we give asymptotically optimal trade–offs between the dilation and the compactness for some special classes of graphs. We proved the compactness threshold $\Theta(\sqrt{n})$ for the dilation $1.25D-1$ on multiglobe graphs and the same threshold $\Theta(\sqrt{n})$ for the dilation D on planar multiglobe graphs (so called globe graphs). Moreover, for globe graphs nearly-optimal (in the sense of $(1+\varepsilon)D$-bounded for any given constant $\varepsilon>0$) routing is achievable with only a constant number of intervals.

3.1. A lower bound on dilation bounded interval routing

Assume $B\subseteq\{0,\dots,n-1\}$. A set A is called *k-interval representable* (shortly k-I) in the set B if A is a subset of B and there are k cyclic intervals $I_1,\dots,I_k$ such that $(\bigcup_{i=1}^{k} I_i)\cap B=A$.

Assume that the cyclically ordered elements of B are $b_0 \prec b_1 \prec \ldots \prec b_{s-1} \prec b_0$. Then the successor of b_i in B is $b_{i\oplus 1}$. An element a of A, $A \subseteq B$, is called an *isolated* element in A w.r.t. B if its successor in B is not in A; otherwise a is called an *inner* element in A w.r.t. B. It is obvious that if A is k-I w.r.t. B then the number of isolated elements in A is at most k and that there are at least $|A| - k$ inner elements in A.

Lemma 2. *Assume $M = \{a_{i,j} \mid 1 \leqslant i \leqslant t,\ 1 \leqslant j \leqslant s\}$ is the $s \times t$ matrix of distinct elements from $\{0, \ldots, n-1\}$ such that every column $C_i = \{a_{i,j} \mid 1 \leqslant j \leqslant s\}$ and every row $R_j = \{a_{i,j} | 1 \leqslant i \leqslant t\}$ is k-I in M. Then the following inequality holds*

$$k \geqslant \frac{st}{s+t}.$$

Proof. Let P be the number of isolated elements in sets $R_1, \ldots, R_s$ w.r.t. M. In every k-I set there are at most k isolated elements, so we have

$$P \leqslant sk.$$

Similarly, there are at least $t(s-k)$ inner elements in sets $C_1, \ldots, C_t$ and one can observe that each of them is isolated in its row. Thus it follows

$$P \geqslant t(s-k).$$

Combining both inequalities we get

$$k \geqslant \frac{st}{s+t}. \qquad \square$$

Further, we construct a graph $F(s,t,r)$ such that due to the Lemma 2 each interval routing scheme on F with the dilation bounded by $1.5D-3$ requires at least $st/(s+t)$ intervals.

Graph $F(s,t,r)$ is defined as follows. There are st "middle" vertices $\{a_{i,j}\}$ which form $s \times t$ rectangle, t "column" vertices $\{c_i\}$, s "row" vertices $\{b_j\}$, and two special vertices b, c. A column vertex c_i is connected with any vertex from the ith column of the rectangle via unique path of the length r and a row vertex b_j is connected with any vertex from the jth row of the rectangle via unique path of the length r. The vertex c is connected with all "column" vertices c_i and the vertex b is connected with all "row" vertices b_j. As an example graph $F(3,3,2)$ is shown in Fig. 5.

Graph $F(s,t,r)$ has $(2r-1)st+s+t+2$ vertices and $2str+s+t$ edges. Its diameter is $2r+2$.

Theorem 6. *There is a graph F such that each $(k, 1.5D{-}3)$-IRS of F needs $k = \Omega(\sqrt{n})$.*

Proof. Assume that $\rho = (\lambda, R)$ is $(k,\ 1.5D-3)$-IRS of $F(s,t,r)$. As ρ is $(1.5D-3)$-bounded, for all columns it holds $\{a_{i,j} \mid 1 \leqslant j \leqslant s\} \subseteq R(c,(c,c_i))$ and for all rows it holds $\{a_{i,j} | 1 \leqslant i \leqslant t\} \subseteq R(b,(b,b_j))$. Otherwise the length of the routing path from c to $a_{i,j}$ and from b to $a_{i,j}$ would be at least $3r{+}1$, thus longer than $1.5D{-}3$. Now, take $s{\times}v$

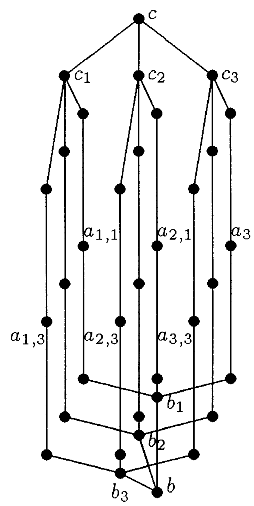

Fig. 5. Graph $F(3,3,2)$.

matrix $M = \{\lambda(a_{i,j}) \mid 1 \leqslant i \leqslant t, 1 \leqslant j \leqslant s\} \subseteq \{0,\dots,n-1\}$, where $n = (2r-1)st+s+t+2$; with columns $C_i = \{\lambda(a_{i,j}) \mid 1 \leqslant j \leqslant s\}$ and rows $R_j = \{\lambda(a_{i,j}) \mid 1 \leqslant i \leqslant t\}$, which all are k-I in M. Applying Lemma 2, we get $k \geqslant st/(s+t)$. Choosing $s = t = 2(k+1)$ we get a contradiction. Hence, there does not exist $(1.5D-3)$-bounded k-IRS of the graph $F(2k+1, 2k+1, r)$ and therefore the claim of the theorem holds. □

Proposition 4. *There is* $(2, 1.5D)$*-IRS of* $F(s,t,r)$*.*

This can be proved using b and c as cluster centers in the proof of the Theorem 7. The resulting scheme uses only 2 intervals.

3.2. An upper bound on the dilation bounded interval routing

In this subsection we show that every graph has interval routing with the dilation $\lceil 1.5D \rceil$ using only $O(\sqrt{n \log n})$ intervals. We need two following lemmas.

Lemma 3. *For* $n \in N$ *and* $m \geqslant \sqrt{n \ln n}$ *it holds* $\binom{n-m}{m} < \binom{n}{m}/n$*.*

Proof. For $m \geqslant \sqrt{n \ln n}$ it holds

$$\frac{n^{\underline{m}}}{(n-m)^{\underline{m}}} \geqslant \left(\frac{n}{n-m}\right)^m = \left(\left(1 + \frac{m}{n-m}\right)^{n/m}\right)^{m^2/n} > e^{m^2/n} \geqslant n$$

where $n^{\underline{m}}$ stands for $n(n-1)\dots(n-m+1)$. Multiplying both sides of the inequality by $\binom{n-m}{m}/n$ we obtain the claim of the lemma. □

Lemma 4. *Let* $G=(V,E)$ *be a graph. There is a set* $C\subseteq V$ *such that* $|C|=\mathrm{O}(\sqrt{n\log n})$ *and for each* $v\in V$ *it holds* $d(v,C)\leqslant\lceil\frac{1}{2}D\rceil$.

Proof. Let $V=\{1,\dots,n\}$ and $m=\lceil\sqrt{n\ln n}\rceil$. It holds $\binom{n-m}{m}<\binom{n}{m}/n$. For a vertex $v\in V$ define the set $V_v\subseteq V$ as a set of vertices whose distance from v is at most $\lceil\frac{1}{2}D\rceil$. If there exists $v\in V$ such that $|V_v|\leqslant m$, then it is obvious that we can set $C=V_v$ and the lemma holds. If such v does not exist (i.e., for all $v\in V$ it holds $|V_v|>m$) we prove the lemma by contradiction. Suppose that the lemma does not hold. If we take the union of any m sets from $V_1,\dots,V_n$, then at least one element from V is not contained in this union. There are $\binom{n}{m}$ possibilities how to choose these m sets and from the pigeon-hole principle follows that there exists $a\in V$ such that a is missing in at least $\binom{n}{m}/n$ choices. On the other hand $|V_a|>m$, therefore a is not contained in at most $n-m$ sets and the number of choices with a missing is at most $\binom{n-m}{m}$. From this we get inequality $\binom{n-m}{m}\geqslant\binom{n}{m}/n$, which is a contradiction. □

Theorem 7. *Let* $G=(V,E)$ *be a graph. There is an interval routing scheme of* G *with the dilation* $\lceil 1.5D\rceil$ *using only* $\mathrm{O}(\sqrt{n\log n})$ *intervals.*

Proof. Take the set $C=\{c_1,\dots,c_m\}\subseteq V$ from the previous lemma. Divide the set V into non-intersecting subsets $R_1,\dots,R_m$ such that for any vertex $v\in R_i$ it holds $d(c_i,v)\leqslant\lceil\frac{1}{2}D\rceil$ and the subgraph of G induced by R_i (denoted as G/R_i) is connected for all $i\in\{1,\dots,m\}$. Subgraphs G/R_i are called clusters and vertices c_i cluster centers. Given the set C we can find this division as follows. Set $\forall i\in\{1,\dots,m\}:R_i:=\{c_i\}$. Then repeat $\lceil\frac{1}{2}D\rceil$ times: for each $i\in\{1,\dots,m\}$ set $R_i:=R_i\cup\{\text{free vertices adjacent to } R_i\}$.

Construct a BFS spanning tree T_i of all G from each center $c_i\in C$. First, create a tree-labeling scheme on the subtree T_i/R_i from the root c_i following the technique from [17] (two intervals per arc are required). Vertices in R_i will have consecutive labels for all $i\in\{1,\dots,m\}$. Then, assign an interval corresponding to R_i to each arc of T_i not belonging to the cluster G/R_i and oriented towards the center c_i. Such interval routing scheme has compactness at most $m+1$ (because each arc belongs to at most m trees, in $m-1$ trees it is assigned 1 interval and in one tree it is assigned two intervals). The dilation is at most $D+\lceil D/2\rceil=\lceil 1.5D\rceil$. □

3.3. Compactness versus dilation for multiglobe graphs

The *multiglobe graph* (denoted as $M(s,t,r)$) is obtained from the complete bipartite graph $K_{s,t}$ by replacing all edges by unique paths of the length r (see Fig. 6 for example). Hence, $K_{s,t}\equiv M(s,t,1)$. The multiglobe graph was introduced in [18]. Its diameter is $2r$, it has $(r-1)st+s+t$ vertices and rst edges.

Let vertices from one bipartition be $u_1,\dots,u_s$ and from the other be $v_1,\dots,v_t$. Take r even and the vertex in the middle of the path from u_i to v_j denote as $a_{i,j}$.

The complete bipartite graph $M(s,t,1)$ can be shortest path routed with 1 interval [21]. By applying Lemma 2 on multiglobe graphs we get the following proposition.

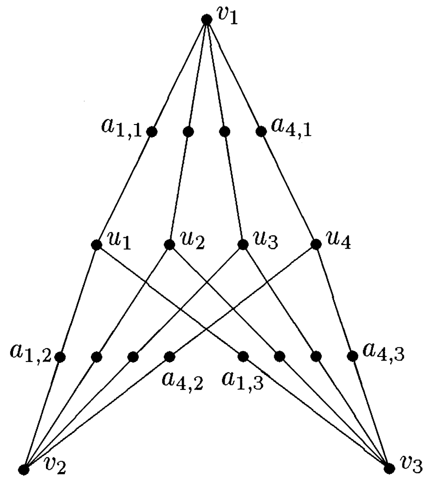

Fig. 6. Multiglobe $M(4, 3, 2)$.

Theorem 8. *There is a multiglobe graph M such that each k-IRS of M with the dilation bounded by* $1.25D - 1$ *needs* $k = \Omega(\sqrt{n})$.

Proof. We will show that for a multiglobe graph $M(s, t, r)$, $r \geqslant 2$, every k-IRS with the dilation bounded by $1.25D - 1$ requires $st/(s + t)$ intervals. Consider a matrix $A = \{\lambda(a_{i,j}) | 1 \leqslant i \leqslant s, 1 \leqslant j \leqslant t\}$. Its rows and columns are k-I in A, because in $(1.25D - 1)$-bounded routing a vertex $a_{i,j}$ is routed along the first arc on the path from u_1 to v_j and also along the first arc on the path from v_1 to u_i. As conditions of Lemma 2 are satisfied, the bound follows from

$$k \geqslant \frac{st}{s+t}. \qquad \square$$

Theorem 9. *There is a* $1.25D$*-bounded* 2*-IRS of the multiglobe graph* $M(s, t, r)$.

Proof. Divide the $M(s, t, r)$ graph into $s + t$ disjoint clusters with cluster centers $v_1, \ldots, v_t, u_1, \ldots, u_s$ and the radius at most $D/4$. Label clusters in the order as their centers appear in the previous sentence, following [17]. Notice that vertices in the union of clusters with centers $v_1, \ldots, v_t$ $(u_1, \ldots, u_s)$ form an interval.

Let e be an arc from the cluster with the center v_i and oriented towards the vertex u_j. One interval on e is used for routing messages to vertices from the cluster v_i reachable from v_i via this arc. Second interval is used for vertices from the cluster u_j. In the case $j = 1$ labels of vertices in clusters $v_{i+1}, \ldots, v_t$ are added to this interval, in the case $j = t$ labels of vertices in clusters $v_1, \ldots, v_{i-1}$ are added to this interval. Any arc f from the cluster v_i oriented towards v_i will be labeled by the complement of the label of the arc emanating from the same vertex as f. Labeling of the arcs in clusters with centers $u_1, \ldots, u_s$ is symmetric. The resulting routing scheme has the dilation bounded by $1.25D$ and uses only two intervals. $\square$

Theorem 10. *For any $\varepsilon > 0$ there exists a k-IRS of the $M(s,t,r)$ with the dilation bounded by $(1+\varepsilon)D$ and $k = \lceil 1/2\varepsilon \rceil \min(s,t)$.*

Proof. W.l.o.g. suppose that $t = \min(s,t)$. Call v_i-flap a tree rooted at v_i with height $r-1$ for $i \neq 1$ and with height r for $i = 1$. Divide each flap into strips of height $\lfloor 2\varepsilon r \rfloor$. The number of strips in any flap is bounded from above by $\lceil 1/2\varepsilon \rceil$. Label the vertices by numbers $1, \ldots, |V|$ in the following order: for each flap take its strips in top-down manner, in every strip take its paths in left-right manner and label each path in top-down manner.

Any arc e emanating from any vertex u_i directs to a different flap. Labels in each flap form one interval, hence at most one interval is needed on this arc. Along any arc e outgoing from any vertex v_i messages to vertices lying on a path to certain u_j-vertex and to those lying on the paths from this u_j to all other v_k's are routed, which gives at most $\lceil 1/2\varepsilon \rceil t$ intervals. Consider any arc e oriented towards an u_j-vertex. Along this arc messages to vertices lying on the path towards u_j, to vertices lying on the paths from u_j to roots of all other flaps and to certain number of consecutive strips from the top of all other flaps are routed. This requires at most $\lceil 1/2\varepsilon \rceil t$ intervals and the dilation is at most D + the height of the strip which is at most $(1+\varepsilon)D$ because $D = 2r$. Any arc oriented towards an v_i vertex has complementary labeling as some arc oriented towards some u_j vertex thus requiring the same number of intervals. $\square$

Note that for any constant ε, $0 < \varepsilon < 0.25$, there is a (k, α)-IRS of $M(s,t,r)$ for $k = O(\min(s,t))$ and $\alpha = (1+\varepsilon)D$, so the lower bound in Theorem 8 is asymptotically tight.

3.4. Compactness versus dilation for globe graphs

The globe graph (denoted as G_s^r, r odd) is a planar multiglobe graph $M(s, 2, \lceil \frac{r}{2} \rceil)$ (see Fig. 7 for example). So it has a vertex set consisting of rs vertices denoted as $a_{i,j}$, which form an $r \times s$ rectangle and two additional vertices b and c. Vertices in columns are connected to form a line segment graph of the length r and the vertex b (c) is connected to vertices $a_{1,i}(a_{r,i})$ for all $i \in \{1, \ldots, s\}$.

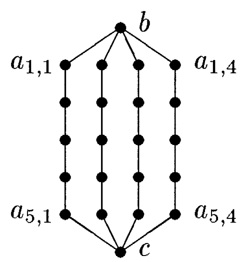

Fig. 7. Globe graph G_4^5.

The class of globe graphs was introduced in [16] and has been studied in connection with the compactness and the dilation in [11, 16, 19]. In [16] it was proved that the shortest path routing on globe graphs cannot be achieved with 1-IRS. In [11] it was shown that the shortest path k-IRS on globe graphs $G_s^{s^2}$ needs $k=\Omega(\sqrt[3]{n})$ intervals. In [19] it was shown that on globe graphs $(k, \frac{2k+1}{2k}D-1)$-bounded IRS needs $k=\Omega(\sqrt[3]{n})$ intervals and $(k, \frac{6k+1}{6k}D-1)$-bounded IRS needs $k=\Omega(\sqrt{n})$ intervals. There is an asymptotically tight trade-off between the compactness and the dilation for globe graphs. Unlike multiglobes, while optimal routing requires a high number of intervals, nearly optimal routing is achievable with only a constant number of intervals.

Theorem 11. *Every optimal IRS of the globe graph G_{s+1}^s needs $s/4$ intervals.*

Proof. Let $\rho=(\lambda,R)$ be an optimal k-IRS on G_{s+1}^s. Assume $A=\{\lambda(a_{i,j}) \mid i,j \in \{1,\dots,s\}\}$. Vertices from the jth column must be routed at the vertex b along the arc $(b,a_{1,j})$, therefore every column $K_j=\{\lambda(a_{i,j}) \mid 1 \leqslant i \leqslant r\}$ is k-I in A. We denote intervals representing the jth column K_j as $I_j^1,\dots,I_j^k$.

For any row index $p \in \{1,\dots,s\}$ let's cut the jth column $\{a_{i,j} \mid 1 \leqslant i \leqslant r\}$ into sets $B_j=\{a_{i,j} \mid i<p\}$ and $C_j=\{a_{i,j} \mid i>p\}$. Every interval has at most two endpoints, therefore for at least $s-2k$ p's we will cut some interval I_j^x such that some of the elements of the set $\{a_{i_1,j},\dots,a_{i_q,j} \mid i_1 < \dots < i_q, \lambda(i_1),\dots,\lambda(i_q) \in I_j^x\}$ will fall into B_j and some into C_j. Such a cut will be called a *good cut*, because if we take interval representations of any sets B and C satisfying $B_j \subseteq B$ and $C_j \subseteq C$, then at least one interval in these representations must end in I_j^x.

There are at least $s(s-2k)$ good cuts in the first s columns and thus there exists a row index p such that at least $s-2k$ columns have a good cut for p.

Now consider arcs $(a_{s+1-p,s}, a_{s-p,s})=e_1$ and $(a_{s+1-p,s}, a_{s+2-p,s})=e_2$. Let B and C be sets of vertices routed from $a_{s+1-p,s}$ along e_1 and e_2, respectively. Vertices $B_j=\{a_{i,j} \mid i<p\}$ must be routed along e_1 and vertices $C_j=\{a_{i,j} \mid i>p\}$ must be routed along e_2 for any $j \in \{1,\dots,s\}$. If we take interval representations of B and C, at least $s-2k$ intervals must end in cut intervals, therefore $k \geqslant \frac{s-2k}{2}$ and from that it follows $k \geqslant \frac{s}{4}$. $\square$

The previous claim is a consequence of a more general result in [19]. We included a new proof of this claim as it is simpler and can be directly exploited in the following note. The graph G_{s+1}^s is a series-parallel graph requiring $\sqrt{n}/4$ intervals. This is an improvement over the previous lower bound in the form $\sqrt{n/72}$ from [19]. The graph P_s^r is obtained from the globe graph G_s^r by replacing vertices b and c by paths $b_1,\dots,b_s$ and $c_1,\dots,c_s$ and connecting b_j with $a_{1,j}$ and c_j with $a_{r,j}$ (see Fig. 8 for an example). The graph P_{s+1}^s is the first example of a bounded degree series-parallel graph of compactness $\Omega(\sqrt{n})$. P_{s+1}^s is also another example (see [9]) of a bounded degree planar graph requiring $\sqrt{n}/4$ intervals. However, the question of expressing the exact compactness for bounded degree planar graphs is still elusive.

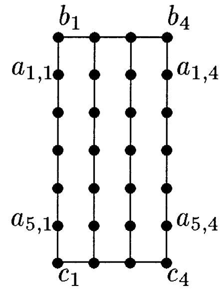

Fig. 8. Graph P_4^5 with similar optimal routing properties as G_4^5.

Theorem 12. *There is an optimal IRS of G_s^r with* $\min(s,r)$ *intervals.*

Proof. Along the arc e from $a_{i,j}$ oriented towards b, messages to vertices $b, a_{1,j}, \ldots, a_{i-1,j}$ and $a_{1,k}, \ldots, a_{r+1-i,k}$ for all $k \neq j$ should be routed. Along the arc e from c to $a_{r,i}$ messages to vertices $a_{1,i}, \ldots, a_{r,i}$ should be routed. For other arcs the situation is symmetric. If we label vertices in order $b, a_{1,1}, \ldots, a_{1,s}, \ldots, a_{r,s}, c$ then r intervals will suffice. If we label them in order $b, a_{1,1}, \ldots, a_{r,1}, \ldots, a_{r,s}, c$ then s intervals will suffice. □

Theorem 13. *There is* $(1, 1.5D)$*-IRS of* G_s^r.

Proof. Label b by 1 and c by $sr+2$. Then label column 1 top down, next column 2 top down, etc. using labels 2 up to $sr+1$ in order. Consider that the vertex b belongs to the first column and the vertex c belongs to the last column. Arcs $(b, a_{1,j})$ and $(c, a_{r,j})$ will be assigned interval corresponding to the jth column for all $j \in \{1, \ldots, s\}$. From some vertex $a_{i,j}$ to a vertex in the same column, routing path follows shortest path, from some vertex $a_{i,j}$ to a vertex in other column, routing path approaches b or c, depending on which is closer to $a_{i,j}$. This can be arranged using only one cyclic interval and the resulting routing is $1.5D$-bounded. □

As a consequence of Theorem 10 or of Theorem 13 in [19] it holds for globe graphs:

Corollary 1. *For arbitrary $\varepsilon > 0$ there is $(1+\varepsilon)D$-bounded IRS of G_s^r with a constant number of intervals.*

3.5. Summary

Table 2 summarizes results from Section 3 on bounds for the number of intervals needed for IRS with the bounded dilation.

4. Conclusion

We proved that large compactness is needed for optimal interval routing on certain regular and symmetric topologies used in parallel architectures. The main question

Table 2

Graphs	Dilation	Compactness
arbitrary graphs [17]	$2D$	1
arbitrary graphs	$1.5D$	$O(\sqrt{n \log n})$
F	$1.5D$	2
F	$\leqslant 1.5D - 3$	$\Omega(\sqrt{n})$
Multiglobe [18]	$\leqslant 1.75D - 1$	> 1
Multiglobe	$1.25D$	2
Multiglobe	$\leqslant 1.25D - 1$	$\Omega(\sqrt{n})$
Globe	$1.5D$	1
Globe [16]	$\leqslant 1.5D - 1$	> 1
Globe [19]	$(1 + \varepsilon)D$	$O(\frac{1}{\varepsilon})$
Globe [19]	D	$\Omega(\sqrt{n})$

remains whether this phenomenon holds also for nearly-optimal interval routing on these topologies.

We also improved a lower bound on compactness for the dilation bounded interval routing on general n-vertex graphs.[2] The complementary upper bound shows that for interval routing with the dilation $\lceil 1.5D \rceil$ the compactness is $O(\sqrt{n \log n})$. The main unresolved problem is to exhibit a tight trade-off between the dilation and the compactness for general graphs.

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